MATH 245 F23, Exam 2 Solutions

1. Carefully define the following terms: Big Omega (Ω), Big Theta (Θ)

   Let a_n, b_n be sequences. We say that a_n is big Omega of b_n if there is some real M and some natural n_0 such that for every $n \geq n_0$ we have $M|a_n| \geq |b_n|$. Let a_n, b_n be sequences. a_n is big Theta of b_n if a_n is big O of b_n, AND a_n is big Omega of b_n.

2. Carefully state the following theorems: Proof by Contradiction Theorem, Proof by Minimum Element Induction Theorem

   The Proof by Contradiction Theorem says: For any propositions p, q, to prove implication $p \to q$, we prove $p \wedge \neg q \equiv F$. The Proof by Minimum Element Induction Theorem says: If a nonempty set of integers has a lower bound, then it has a minimum.

3. Let a_n, b_n, c_n be sequences of real numbers. Suppose that $a_n = O(c_n)$ and $b_n = O(c_n)$. Set $d_n = a_n + b_n$. Prove that $d_n = O(c_n)$.

   Because $a_n = O(c_n)$, there are $M_a \in \mathbb{R}$ and $n_a \in \mathbb{N}$ such that if $n \geq n_a$ then $|a_n| \leq M_a|c_n|$. Because $b_n = O(c_n)$, there are $M_b \in \mathbb{R}$ and $n_b \in \mathbb{N}$ such that if $n \geq n_b$ then $|b_n| \leq M_b|c_n|$. We need these four constants M_a, M_b, n_a, n_b to find M_d, n_d.

   Let $M = \max(M_a, M_b)$, $M_d = 2M$, and $n_d = \max(n_a, n_b)$. Let $n \geq n_d$. Note that $n \geq n_a$, so $|a_n| \leq M_a|c_n| \leq M|c_n|$. Note also that $n \geq n_b$, so $|b_n| \leq M_b|c_n| \leq M|c_n|$. Finally, we have $|d_n| = |a_n + b_n| \leq |a_n| + |b_n| \leq M|c_n| + M|c_n| = 2M|c_n| = M_d|c_n|$.

   Note: $|x+y| \leq |x|+|y|$ by the triangle inequality. It is not correct to say $|x+y| = |x|+|y|$ unless we know that x, y are each positive (which we don't here). However I did not take points off for this error.

4. Prove that $\forall x \in \mathbb{R},\ 5x - 3|x+2| < 2x - 5$.

   Let $x \in \mathbb{R}$. We have two cases, based on whether or not $x + 2 \geq 0$ (i.e. $x \geq -2$).

   Case $x \geq -2$: Now $|x+2| = x+2$, so $5x - 3|x+2| = 5x - 3(x+2) = 2x - 6 < 2x - 5$.
   Case $x < -2$: Now $|x+2| = -(x+2)$, so $5x - 3|x+2| = 5x + 3(x+2) = 8x + 6$. Since $x < -2$ in this case, we multiply by 6 to get $6x < -12 < -11$. Add $2x + 6$ to both sides to get $8x + 6 < 2x - 5$. Combining with the previous, we get $5x - 3|x+2| < 2x - 5$.

   In both cases, the desired result $5x - 3|x+2| < 2x - 5$ holds.

5. Prove or disprove: $\forall x \in \mathbb{R},\ \lceil x \lfloor x \rfloor \rceil = \lfloor x \lfloor x \rfloor \rfloor$.

   The statement is false, and requires an explicit counterexample. Many solutions are possible. One solution is: Take $x^\star = 1.2$. We have $\lfloor x^\star \rfloor = 1$ and $x^\star \lfloor x^\star \rfloor = 1.2$. Hence $\lceil x^\star \lfloor x^\star \rfloor \rceil = 2$ while $\lfloor x^\star \lfloor x^\star \rfloor \rfloor = 1$.

6. Prove that $\forall n \in \mathbb{N},\ 4^n > 3^n$.

   This is proved by (vanilla) induction. Base case $n = 1$: $4^1 = 4 > 3 = 3^1$.
   Inductive case: Let $n \in \mathbb{N}$ and assume that $4^n > 3^n$. Multiply both sides by 4 to get $4^{n+1} = 4 \cdot 4^n > 4 \cdot 3^n \geq 3 \cdot 3^n = 3^{n+1}$. Hence $4^{n+1} > 3^{n+1}$.

7. Solve the recurrence that has initial conditions $a_0 = 2, a_1 = 7$ and relation $a_n = a_{n-1} + 2a_{n-2}$ $(n \geq 2)$.

The characteristic polynomial is $r^2 - r - 2 = (r-2)(r+1)$, which has roots $2, -1$. Hence the general solution is $a_n = A2^n + B(-1)^n$. We now apply the initial conditions $2 = a_0 = A2^0 + B(-1)^0 = A + B$, $7 = a_1 = A2^1 + B(-1)^1 = 2A - B$. Solving the system $\{2 = A + B, 7 = 2A - B\}$ we get $A = 3, B = -1$. Hence the desired specific solution is $a_n = 3 \cdot 2^n - (-1)^n$. If desired, this can be rewritten/simplified to $a_n = 3 \cdot 2^n + (-1)^{n+1}$.

WARNING: $3 \cdot 2^n = 3\,(2^n) \neq (3 \cdot 2)^n = 6^n$.

8. Prove $\forall n \in \mathbb{N}_0 \; !m \in \mathbb{N}_0, \; m^3 \leq n < (m+1)^3$.

Let $n, m_1, m_2 \in \mathbb{N}_0$ be arbitrary. Suppose that $m_1^3 \leq n < (m_1+1)^3$ and $m_2^3 \leq n < (m_2+1)^3$. We recombine to get $m_1^3 \leq n < (m_2+1)^3$, hence $m_1^3 < (m_2+1)^3$. Applying cube roots to both sides we get $m_1 < m_2 + 1$. Starting over, we recombine again to get $m_2^3 \leq n < (m_1+1)^3$, hence $m_2^3 < (m_1+1)^3$. Applying cube roots to both sides we get $m_2 < m_1 + 1$. Subtracting one and combining, we get $m_2 - 1 < m_1 < m_2 + 1$. Since m_1, m_2 are integers, we use a theorem from the book (Thm 1.12(d)) to get $m_1 = m_2$.

ALTERNATE PROOF: Let $m, m_1, m_2 \in \mathbb{N}_0$ be arbitrary. Suppose that $m_1^3 \leq n < (m_1+1)^3$ and $m_2^3 \leq n < (m_2+1)^3$. Now, we take cube roots of the first equation to get $m_1 \leq \sqrt[3]{n} < m_1 + 1$. But also $\lfloor \sqrt[3]{n} \rfloor \leq \sqrt[3]{n} \leq \lfloor \sqrt[3]{n} \rfloor + 1$. We have two integers, m_1 and $\lfloor \sqrt[3]{n} \rfloor$, satisfying the same double inequality. By the uniqueness of floor, we must have $m_1 = \lfloor \sqrt[3]{n} \rfloor$. We start over, taking cube roots of the second equation to get $m_2 \leq \sqrt[3]{n} < 2_1 + 1$. Again we apply uniqueness of floor to get $m_2 = \lfloor \sqrt[3]{n} \rfloor$. Hence $m_1 = \lfloor \sqrt[3]{n} \rfloor = m_2$, so we conclude $m_1 = m_2$.

9. Use maximum element induction to prove $\forall n \in \mathbb{N}_0 \; \exists m \in \mathbb{N}_0, \; m^3 \leq n < (m+1)^3$.

Let $n \in \mathbb{N}_0$ be arbitrary, and set $S = \{a \in \mathbb{N}_0 : a^3 \leq n\}$ or $S = \{a \in \mathbb{Z} : a \geq 0 \wedge a^3 \leq n\}$. Note that S is nonempty, because $0 \in S$ (since $0^3 = 0 \leq n$). Also note that $\sqrt[3]{n}$ is an upper bound for S, since if $a \in S$ then $a^3 \leq n$ and hence $a \leq \sqrt[3]{n}$. Maximum element induction gives us a maximum $m \in S$, i.e. $m^3 \leq n$ but $(m+1)^3 \not\leq n$. Combining, we get $m^3 \leq n < (m+1)^3$.

NOTE: If you use $S = \{a \in \mathbb{Z} : a^3 \leq n\}$, then you can still do maximum element induction (it's easier to prove that S is nonempty, since it's a halfline) to find $m \in \mathbb{Z}$ with $m^3 \leq n < (m+1)^3$, but you now have to worry about whether $m \in \mathbb{N}_0$ or not.

10. Prove that for all $n \in \mathbb{Z}$ with $n \geq 3$, that $F_n \leq 3F_{n-2}$. Here F_n denotes the Fibonacci numbers.

We need strong induction and two base cases: $n = 3$ has $F_3 = 2 \leq 3 = 3F_1$, and $n = 4$ has $F_4 = 3 \leq 3 = 3F_2$.

Inductive case: Let $n \in \mathbb{Z}$ with $n \geq 5$, and suppose that the predicate is true for all smaller n (that are at least 3). In particular, it is true for $n-1$ and $n-2$. Hence $F_{n-1} \leq 3F_{n-3}$ and $F_{n-2} \leq 3F_{n-4}$. We add these inequalities, getting $F_{n-1} + F_{n-2} \leq 3F_{n-3} + 3F_{n-4} = 3(F_{n-3} + F_{n-4})$. Now, the defining recurrence of Fibonacci numbers gives $F_n = F_{n-1} + F_{n-2}$ and $F_{n-2} = F_{n-3} + F_{n-4}$. Substituting, we get $F_n \leq 3F_{n-2}$.